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On automorphisms of $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$

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Abstract

We will study the automorphisms on the group II_1 factor $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ which preserve the subalgebra $L(SL(2,\mathbb{Z}))$. Our main result is

$$\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z}))) \simeq \mathbb{Z}_{12} \rtimes \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts on $\mathbb{Z}_{12}$ by the inverse operation. The proof is a modification of the recent paper due to Neshveyev and Størmer for non-commutative groups. The uniqueness of HT-Cartan subalgebras due to Popa plays a crucial role in the proof.

Keywords: von Neumann algebras; Automorphisms; HT-Cartan subalgebras; $SL(2,\mathbb{Z})$

1. Introduction

The rigidity result of von Neumann algebras first appeared in Connes' paper [1] in which II_1 factors with countable fundamental groups were constructed by using discrete groups with property T of Kazhdan [7]. The concept of property T was then extended to all von Neumann algebras by Connes and Jones [2] and after that, Popa [9] systematically investigated the rigidity for the inclusions of II_1 factors and got some rigidity results of inclusions. Moreover, recently Popa obtained in [10] the first example of II_1 factors with trivial fundamental groups by using the rigidity of inclusions, with the help of Haagerup property [5].

In this paper, we will show the rigidity result on automorphisms of $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$: We shall determine the structure of all automorphisms on

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$L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ which preserve the subalgebra $L(SL(2,\mathbb{Z}))$ globally. The set of these automorphisms is denoted by $\mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))$, and we write

$$\mathrm{Int}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z}))) = \{\mathrm{Ad}\, w : w \text{ is a unitary in } L(SL(2,\mathbb{Z}))\}.$$

Our main result is

$$\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z}))) \simeq \mathbb{Z}_{12} \rtimes \mathbb{Z}_2,$$

where

$$\begin{aligned}&\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))\\ &\quad = \mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))/\mathrm{Int}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))\end{aligned}$$

and $\mathbb{Z}_2$ acts on $\mathbb{Z}_{12}$ by the inverse operation. Indeed, the automorphism group $\mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))$ can be completely described by the irreducible characters and automorphisms on $SL(2,\mathbb{Z})$.

Our strategy, which is very simple, is a modification of the argument in the paper due to Neshveyev and Størmer [8] in the non-commutative group setting. Let us make some explanation about the result due to Neshveyev and Størmer: Let G be a countable abelian group with a weakly mixing free measure-preserving action on the probability space (X,μ). We will denote by α the induced action of G on $L^\infty(X,\mu)$. Let γ be an automorphism on $L^\infty(X,\mu) \rtimes_\alpha G$ satisfying $\gamma(L(G)) = L(G)$. Assume that there exists a unitary $u \in L^\infty(X,\mu) \rtimes_\alpha G$ such that $u^*\gamma(L^\infty(X,\mu))u = L^\infty(X,\mu)$. Then, Neshveyev and Størmer showed that there exist a unitary $w \in L(G)$, a character χ on G, an automorphism β on G and a measure-preserving transformation S on X which satisfies $Sg = \beta(g)S$ for $g \in G$, such that $\gamma = \mathrm{Ad}\, w\sigma_S\sigma_\chi$. (See Section 2 for the definition of σ_S, σ_χ.) We will show this theorem for the non-commutative group $SL(2,\mathbb{Z})$ acting on $\mathbb{T}^2$. As the commutativity of the group is used in crucial parts of the proof of [8], we have to make some effort to fit their argument to the non-commutative group. It is important that the unitary conjugacy-assumption in the above theorem is not needed in our setting. Indeed for $\gamma \in \mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})))$, we can always find a unitary such that $\mathrm{Ad}\, u^*\gamma(L(\mathbb{Z}^2)) = L(\mathbb{Z}^2)$, thanks to the uniqueness of HT-Cartan subalgebras established by Popa [10]. This fact is crucial in our proof. We would like to note that in the most part of our proof, we do not use the special structure of $SL(2,\mathbb{Z})$, i.e. our proof does work for more general non-commutative groups. However, if we replace $SL(2,\mathbb{Z})$ by some other groups, the assumption of the unitary conjugacy for Cartan subalgebras is needed and hence we cannot get the complete structure of such automorphisms. Thus, we restrict our attention to this special case, where we can use Popa's theorem.

2. Main result

At first, we would like to fix some notations.

The unimodular group $SL(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ by the matrix multiplication. Then its dual action on $\hat{\mathbb{Z}}^2=\mathbb{T}^2$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}(z,w)=(z^a w^c, z^b w^d)$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}).$$

We shall freely identify these two actions via the Fourier transformation and this identification induces the natural isomorphism between $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ and $L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$, where α denotes the action of $SL(2,\mathbb{Z})$ on $L^\infty(\mathbb{T}^2)$ induced by this action.

For each automorphism β on $SL(2,\mathbb{Z})$, consider all measure-preserving transformations S on $\mathbb{T}^2$ such that $Sg=\beta(g)S$ for $g\in SL(2,\mathbb{Z})$. We denote by I_β the set of all such transformations S. A measure-preserving transformation T on $\mathbb{T}^2$ induces the automorphism σ_T defined by $\sigma_T(f)(x)=f\circ T^{-1}(x)$ $(f\in L^\infty(\mathbb{T}^2),\, x\in\mathbb{T}^2)$. For $S\in I_\beta$, the automorphism σ_S can be extended to $L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$ by $\sigma_S(\lambda_g)=\lambda_{\beta(g)}$, where λ_g is the canonical implementing unitary. An irreducible character χ on $SL(2,\mathbb{Z})$ also gives the automorphism σ_χ on $L(\mathbb{Z}^2\rtimes SL(2,\mathbb{Z}))$ such that $\sigma_\chi(\lambda_g)=\chi(g)\lambda_g$ and $\sigma_\chi|_{L(\mathbb{Z}^2)}=\mathrm{id}$.

The following theorem is an analogue of [8, Theorem 4.2] for the non-commutative group $SL(2,\mathbb{Z})$.

Theorem 2.1. *Let γ be an automorphism of $L(\mathbb{Z}^2\rtimes SL(2,\mathbb{Z}))$ satisfying $\gamma(L(SL(2,\mathbb{Z})))=L(SL(2,\mathbb{Z}))$. Then there exist a unitary $w\in L(SL(2,\mathbb{Z}))$, an irreducible character χ on $SL(2,\mathbb{Z})$, an automorphism β on $SL(2,\mathbb{Z})$ and a transformation $S\in I_\beta$ such that*

$$\gamma=\mathrm{Ad}\, w\sigma_S\sigma_\chi.$$

Corollary 2.2. *We have an isomorphism*

$$\mathrm{Out}(L(\mathbb{Z}^2\rtimes SL(2,\mathbb{Z})),L(SL(2,\mathbb{Z})))\simeq\mathbb{Z}_{12}\rtimes\mathbb{Z}_2.$$

Proof. First, we shall show that $\sigma_S\sigma_\chi$ is an outer automorphism on $L(\mathbb{Z}^2\rtimes SL(2,\mathbb{Z}))$ whenever β is outer or χ is a non-trivial character. In order to show this fact, we need the following claim.

Claim. *$L(SL(2,\mathbb{Z}))$ is singular in $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, i.e. if w is a normalizer of $L(SL(2,\mathbb{Z}))$ in $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, then w must belong to $L(SL(2,\mathbb{Z}))$.*

The proof of this claim will be postponed until the end of this section.

If $\sigma_S\sigma_\chi = \mathrm{Ad}\, w$ for some unitary $w \in L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, then w is a normalizer of $L(SL(2,\mathbb{Z}))$ and hence $w \in L(SL(2,\mathbb{Z}))$ by the above claim. Then by the proof of [8, Proposition 2.2], we have $w = c\lambda_g$ for some scalar c and $g \in SL(2,\mathbb{Z})$. (Indeed, this can be easily seen by using the Fourier expansion of w.) Then direct computations show that this can occur only when β is inner and χ is trivial.

Thanks to the above consideration, we have only to prove that the subgroup generated by $\{\sigma_S\}_{S \in I_\beta,\beta}$ and $\{\sigma_\chi\}_\chi$ in $\mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})))$ is isomorphic to $\mathbb{Z}_{12} \rtimes \mathbb{Z}_2$.

It is a well-known fact that $SL(2,\mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ where

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are generators of $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_2$, respectively. Hence, all irreducible characters on $SL(2,\mathbb{Z})$ are of the form $\chi_1 * \chi_2$ for some $\chi_1 \in \hat{\mathbb{Z}}_4$ and $\chi_2 \in \hat{\mathbb{Z}}_6$ which coincide on $\mathbb{Z}_2$. Thus, it is easily seen that the group consisting of all irreducible characters on $SL(2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_{12}$.

It is also well known that up to inner automorphism, the map

$$\beta = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the unique outer automorphism on $SL(2,\mathbb{Z})$ which does not come from some character [6]. Clearly, this map β induces the inverse operation on $\mathbb{Z}_4, \mathbb{Z}_6 (\subset SL(2,\mathbb{Z}))$ and hence on the characters. Define the transformation S on $\mathbb{T}^2$ by $S(z,w) = (z,\bar{w})$. Then direct computations show that $S \in I_\beta$. Note that S (and σ_S) has period 2 and $\sigma_S\sigma_\chi = \sigma_{\chi\circ\beta}\sigma_S$ holds. Golodets [3] showed that I_{id} consists of exactly two elements; identity map and conjugation map. It is easily seen that $S_1^{-1} \cdot S_2 \in I_{id}$ if $S_1, S_2 \in I_\beta$. Since the conjugation map is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(the generator of } \mathbb{Z}_2\text{)},$$

we have the above statement. $\square$

In order to prove the above theorem, we need the uniqueness theorem for HT-Cartan subalgebras due to Popa. More precisely, we need:

Theorem 2.3 (Popa [10, Theorem 4.1]). *Let γ be an automorphism on $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$. Then there exists a unitary $u \in L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ such that $\mathrm{Ad}\, u^*\gamma(L(\mathbb{Z}^2)) = L(\mathbb{Z}^2)$.*

Thanks to this theorem, we are now in the same situation as that of [8]. Unfortunately, Neshveyev and Størmer's proof uses the commutativity of the group frequently, so we cannot apply their argument directly. However their argument does work in our setting with some modifications.

The rest of this section will be devoted to the proof of the main result. As we noted in the introduction, the proof follows the arguments of [8]. We recommend the reader to consult the paper of Neshveyev and Størmer, though our proof is self-contained.

Before starting the proof, we would like to give an outline of the main steps of it. First, we will introduce the unitary W and 1-cocycles $g(h,x)$, $t(h,x)$. It can be shown that they satisfy some equation (Lemma 2.6) by the same argument of [8]. Next we will investigate these cocycles by using this equation and determine their form explicitly. Here is the main difference between our argument and original one [8]. In [8], 2-cocycle vanishing theorem for abelian groups is used in order to determine the cocycles [8, Lemma 4.7]. Since we are dealing with the non-commutative group $SL(2,\mathbb{Z})$, another method will be needed. After having determined the cocycles, we prove Theorem 2.1 by the same method as in [8].

We consider the standard representation of $L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$ on $L^2(\mathbb{T}^2)\otimes l^2(SL(2,\mathbb{Z}))$. Let π be the representation of $L^\infty(\mathbb{T}^2)$ given by

$$\pi(f) = \sum_{g\in SL(2,\mathbb{Z})} \alpha_{g^{-1}}(f)\otimes e_g,$$

where e_g is the minimal projection on $\mathbb{C}\delta_g$ $(g\in SL(2,\mathbb{Z}))$ and $f\in L^\infty(\mathbb{T}^2)$. We sometimes omit the symbol π, so the reader should not confuse $\pi(L^\infty(\mathbb{T}^2))$ with $L^\infty(\mathbb{T}^2)\otimes I$. We denote the left regular representation (resp. the right regular antirepresentation) of $SL(2,\mathbb{Z})$ on $l^2(SL(2,\mathbb{Z}))$ by λ_g (resp. ρ_g) $(g\in SL(2,\mathbb{Z}))$. Thus, $L(SL(2,\mathbb{Z})) = \{\lambda_g\}''_{g\in SL(2,\mathbb{Z})}$ and $L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$ is generated by $\pi(L^\infty(\mathbb{T}^2))$ and $L(SL(2,\mathbb{Z}))$.

The algebras $L^\infty(\mathbb{T}^2)$ and $L(SL(2,\mathbb{Z}))$ act standardly on $L^2(\mathbb{T}^2)$ and $l^2(SL(2,\mathbb{Z}))$, respectively. For each automorphism $\alpha\in \mathrm{Aut}(L^\infty(\mathbb{T}^2))$ (resp. $\alpha'\in\mathrm{Aut}(L(SL(2,\mathbb{Z})))$), we denote its canonical implementing unitary by $u_\alpha\in B(L^2(\mathbb{T}^2))$ (resp. $v_{\alpha'}\in B(l^2(SL(2,\mathbb{Z})))$). For each measure-preserving transformation S on $\mathbb{T}^2$, we write $u_S = u_{\sigma_S}$. We also use the notation $u_g = u_{\alpha_g}$, $v_\chi = v_{\sigma_\chi}$ and $v_\beta = v_{\sigma_S}$ $(S\in I_\beta)$.

The modular conjugation of $L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$ is denoted by J. It is easily seen that $J\pi(f)J = \bar{f}\otimes I$ and $J(I\otimes\lambda_g)J = u_g\otimes\rho_g^*$.

Let γ be as in the theorem and take a unitary $u\in L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$ such that $u^*\gamma(L^\infty(\mathbb{T}^2))u = L^\infty(\mathbb{T}^2)$ (Here we use the uniqueness of HT-Cartan subalgebras.) Let $\tilde{\gamma} = \mathrm{Ad}\,u^*\gamma$. The canonical implementation of γ is given by U_γ and we define $U = Ju^*JU_\gamma$. (Here we note that the canonical implementation commutes with the modular conjugation J. See [4] for more information about the standard form of von Neumann algebras.) Then it is easily seen that $\mathrm{Ad}\,U|_{L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})} = \gamma$ and

$\mathrm{Ad}\, U|_{L^\infty(\mathbb{T}^2)\otimes I} = \tilde{\gamma}\otimes I$. (Remark that in $L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})$, $\tilde{\gamma}$ preserve $\pi(L^\infty(\mathbb{T}^2))$ globally. Hence, we can define the automorphism $\tilde{\gamma}\otimes I$ on $L^\infty(\mathbb{T}^2)\otimes I$). Define $W = U(u_{\tilde{\gamma}}^*\otimes v_\gamma^*)$. Clearly, W belongs to $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))$ (see [8, Section 4]).

Consider the Fourier expansion

$$\tilde{\gamma}^{-1}(\lambda_h) = \sum_{g\in SL(2,\mathbb{Z})} E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)\lambda_g,$$

where E denotes the trace-preserving conditional expectation on $\pi(L^\infty(\mathbb{T}^2))$. Let $f_g^{(h)}$ be the support projection of $E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)$. The next lemma is obvious.

Lemma 2.4. *$f_g^{(h)}\perp f_{g'}^{(h)}$ $(g\neq g')$ and $\sum_{g\in SL(2,\mathbb{Z})} f_g^{(h)} = I$.*

Proof. For $f\in L^\infty(\mathbb{T}^2)$, we have

$$\tilde{\gamma}^{-1}\alpha_h\tilde{\gamma}(f)E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*) = E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)\alpha_g(f)$$

and hence

$$\tilde{\gamma}^{-1}\alpha_h\tilde{\gamma}(f)\Big|_{f_g^{(h)}L^\infty(\mathbb{T}^2)} = \alpha_g(f)\Big|_{f_g^{(h)}L^\infty(\mathbb{T}^2)}.$$

Since $SL(2,\mathbb{Z})$ acts freely on $\mathbb{Z}^2$, we get the first statement. Then the Fourier expansion of $\tilde{\gamma}^{-1}(\lambda_h)$ ensures the second statement. $\square$

For almost all $x\in\mathbb{T}^2$, there exists a unique element $g(h,x)\in SL(2,\mathbb{Z})$ such that $f_{g(h,x)}^{(h)}(x) = 1$ and $f_g^{(h)}(x) = 0$ $(g\neq g(h,x))$. Define $\tilde{g}(h,x) = g(h,\sigma^{-1}x)$ where σ is a measure-preserving transformation corresponding to $\tilde{\gamma}$, i.e., σ satisfies $\tilde{\gamma}(f) = f\circ\sigma^{-1}$ for $f\in L^\infty(\mathbb{T}^2)$.

The following lemma is well-known.

Lemma 2.5. *For almost all $x\in\mathbb{T}^2$, we have $g(h,x)^{-1}x = \sigma^{-1}h^{-1}\sigma x$. The map $g(h,x)$ is a 1-cocycle with respect to $\tilde{\gamma}^{-1}\alpha\tilde{\gamma}$, i.e., $g(h,x)g(k,\sigma^{-1}h^{-1}\sigma x) = g(hk,x)$. (Hence, $\tilde{g}(h,x)$ is a 1-cocycle with respect to α.)*

Proof. Recall that $\tilde{\gamma}^{-1}\alpha_h\tilde{\gamma}(f)|_{f_g^{(h)}L^\infty(\mathbb{T}^2)} = \alpha_g(f)|_{f_g^{(h)}L^\infty(\mathbb{T}^2)}$. The first part of the statement is now obvious from the definition of $g(h,x)$. By the equation $\tilde{\gamma}^{-1}(\lambda_{h_1h_2}) = \tilde{\gamma}^{-1}(\lambda_{h_1})\tilde{\gamma}^{-1}(\lambda_{h_2})$ and the previous lemma, we have $f_g^{(h_1)}f_l^{(h_1h_2)} = f_g^{(h_1)}\alpha_g(f_{g^{-1}l}^{(h_2)})$. Thus, $f_l^{(h_1h_2)}(x) = 1$ if and only if there exists $g\in SL(2,\mathbb{Z})$ such that $f_g^{(h_1)}(x)f_{g^{-1}l}^{(h_2)}(g^{-1}x) = 1$ if and only if $g(h_1,x)^{-1}l = g(h_2, g(h_1,x)^{-1}x)$. Hence, we have $g(h_1,x)^{-1}g(h_1h_2,x) = g(h_2,g(h_1,x)^{-1}x) = g(h_2,\sigma^{-1}h_1^{-1}\sigma x)$. $\square$

The automorphism γ is extended to $R(SL(2,\mathbb{Z}))$ by $\mathrm{Ad}\, v_\gamma$. Recall that the unitary W belongs to $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z})) = L^\infty(\mathbb{T}^2,\ R(SL(2,\mathbb{Z})))$. Hence for $x\in\mathbb{T}^2$, we consider that $W(x)$ is a unitary in $R(SL(2,\mathbb{Z}))$.

Lemma 2.6. $W(h^{-1}x) = t(h,x)\rho_h W(x)\gamma(\rho^*_{\tilde{g}(h,x)})$, *where* $t(h,x) = E(\tilde{\gamma}(\lambda_{\tilde{g}(h,x)})\lambda_h^*)(x)$.

Proof. Thanks to Lemma 2.4, we have

$$f_g^{(h)}\tilde{\gamma}^{-1}(\lambda_h) = E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)\lambda_g.$$

Since $\tilde{\gamma} = \mathrm{Ad}\, u^* U_\gamma$,

$$f_g^{(h)} U_\gamma^* u\lambda_h = E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)\lambda_g U_\gamma^* u.$$

Applying $\mathrm{Ad}\, J$ to both sides, we have

$$\left(f_g^{(h)}\otimes I\right)U_\gamma^* JuJ(u_h\otimes\rho_h^*) = (E(\lambda_g\tilde{\gamma}^{-1}(\lambda_h^*))\otimes I)(u_g\otimes\rho_g^*)U_\gamma^* JuJ.$$

Using the equation $W = Ju^*JU_\gamma(u_{\tilde{\gamma}}^*\otimes v_\gamma^*)$, we get

$$\left(\tilde{\gamma}\left(f_g^{(h)}\right)\otimes I\right)W^*(u_h\otimes\rho_h^*) = (E(\tilde{\gamma}(\lambda_g)\lambda_h^*)\otimes I)(u_{\tilde{\gamma}g\tilde{\gamma}^{-1}}\otimes\gamma(\rho_g^*))W^*.$$

Because of the equation $E(\tilde{\gamma}(\lambda_g)\lambda_h^*)u_{\tilde{\gamma}g\tilde{\gamma}^{-1}} = E(\tilde{\gamma}(\lambda_g)\lambda_h^*)u_h$, we see that

$$(\tilde{\gamma}(f_g^{(h)})\otimes I)W^*(I\otimes\rho_h^*) = (E(\tilde{\gamma}(\lambda_g)\lambda_h^*)\otimes I)(I\otimes\gamma(\rho_g^*))\alpha_h\otimes id(W^*).$$

(The equation $E(\tilde{\gamma}(\lambda_g)\lambda_h^*)u_{\tilde{\gamma}g\tilde{\gamma}^{-1}} = E(\tilde{\gamma}(\lambda_g)\lambda_h^*)u_h$ is obtained as follows: As it was seen in the proof of Lemma 2.4, we have $E(\tilde{\gamma}(\lambda_g)\lambda_h^*)\tilde{\gamma}\alpha_g\tilde{\gamma}^{-1} = E(\tilde{\gamma}(\lambda_g)\lambda_h^*)\alpha_h$. This fact and the definition of canonical implementing unitaries show the desired equation.) All elements in the above equation belong to $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))$. Hence, we can consider the evaluations at $x\in\mathbb{T}^2$ and get

$$W(h^{-1}x) = t(h,x)\rho_h W(x)\gamma(\rho^*_{\tilde{g}(h,x)}). \qquad \square$$

Lemma 2.7. *Denote the comultiplication on* $R(SL(2,\mathbb{Z}))$ *by* Δ, *which is defined as* $\Delta(\rho_g) = \rho_g\otimes\rho_g$. *Then we have*

$$F(h^{-1}x) = t(h,x)\Phi(h)^* F(x)\Psi(h),$$

where F, Φ *and* Ψ *are defined by* $F(x) = \gamma^{-1}(W(x))\otimes\gamma^{-1}(W(x))\Delta\circ\gamma^{-1}(W(x))^*$, $\Phi(h) = \gamma^{-1}(\rho_h)^*\otimes\gamma^{-1}(\rho_h)^*$ *and* $\Psi(h) = \Delta\circ\gamma^{-1}(\rho_h)^*$. *Note that* Φ *and* Ψ *are unitary representations of* $SL(2,\mathbb{Z})$ *and* F *is a unitary of* $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$.

Proof. Applying $\Delta\circ\gamma^{-1}$ to both sides of the equation in the previous lemma, we have

$$\Delta\circ\gamma^{-1}(W(h^{-1}x)) = t(h,x)\Delta\circ\gamma^{-1}(\rho_h)\Delta\circ\gamma^{-1}(W(x))\left(\rho^*_{\tilde{g}(h,x)}\otimes\rho^*_{\tilde{g}(h,x)}\right),$$

while by taking a tensor product and applying $\gamma^{-1}\otimes\gamma^{-1}$,

$$\begin{aligned}&\gamma^{-1}(W(h^{-1}x))\otimes\gamma^{-1}(W(h^{-1}x))\\&\quad = t(h,x)^2(\gamma^{-1}(\rho_h)\otimes\gamma^{-1}(\rho_h))(\gamma^{-1}(W(x))\otimes\gamma^{-1}(W(x)))(\rho^*_{\tilde{g}(h,x)}\otimes\rho^*_{\tilde{g}(h,x)}).\end{aligned}$$

Combining these equalities, we get

$$F(h^{-1}x) = t(h,x)\Phi(h)^*F(x)\Psi(h). \qquad \square$$

We will use the following well-known fact: there is a sequence $\{h_n\}_{n=1}^{\infty}\subset SL(2,\mathbb{Z})$ which has the properties (1) h_n tends to infinity, (2) for any finite set $\Omega\subset\mathbb{Z}^2$ such that $(0,0)\notin\Omega$, we can find a sufficiently large n_0 such that $h_n\Omega\cap\Omega=\emptyset$ for $n>n_0$. Indeed if we let for example

$$h_n = \begin{pmatrix} n^2-n+1 & n\\ n-1 & 1\end{pmatrix},$$

then it is easy to see that this sequence h_n is the desired one.

Take such $\{h_n\}_{n=1}^{\infty}\subset SL(2,\mathbb{Z})$ and fix it. Recall that the unitary F belongs to $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. The unitaries $\Phi(h)$ and $\Psi(h)$ $(h\in SL(2,\mathbb{Z}))$ belong to $I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. Let $t_h(x)=t(h,x)$. Then t_h is a unitary element in $L^\infty(\mathbb{T}^2)$. The previous lemma means that

$$\alpha_h(F) = t_h\Phi(h)^*F\Psi(h).$$

Lemma 2.8. *The automorphism* $\theta=\operatorname{Ad}F$ *on* $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ *satisfies*

$$\theta(I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))) = I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z})).$$

Proof. By using the previous lemma, it is easily seen that

$$FxF^* = \Phi(h)\alpha_h(F)\Psi(h)^*x\Psi(h)\alpha_h(F)^*\Phi(h)^*$$

for any $x\in I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ $(||x||\leqslant 1)$. In particular, this equality holds for h_n. We can replace F by F_0 such that it has finite support as an element of $L(\mathbb{Z}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. More precisely, by Kaplansky density theorem, for any $\varepsilon>0$ there exists a finite subset $\Omega\subset\mathbb{Z}^2$ such that $F_0=\sum_{g\in\Omega}a_g\delta_g$ (a_g is an element of $R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$) and $||F-F_0||_2<\varepsilon$ and $||F_0||\leqslant||F||$.

Hence we get

$$||F_0xF_0^* - \Phi(h)\alpha_h(F_0)\Psi(h)^*x\Psi(h)\alpha_h(F_0)^*\Phi(h)^*||_2 < 4\varepsilon.$$

As n goes to infinity, the support of $\Phi(h)\alpha_h(F_0)\Psi(h)^*x\Psi(h)\alpha_h(F_0)^*\Phi(h)^*$ goes to infinity except for the unit $(0,0)\in\mathbb{Z}^2$, while the support of $F_0xF_0^*$ does not change. Since ε is arbitrary, this means that the support of FxF^* consists of only one point $(0,0)$. Hence $FxF^*\in I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$.

By the same argument, we can also see that $F^*xF\in I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. Thus we get the statement. $\square$

Let $\Pi(h) = F\Psi(h)^*F^*\Phi(h)$. Then we have $\alpha_h(F) = t_h\Pi(h)^*F$. Thanks to the previous lemma, each $\Pi(h)$ belongs to $I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$.

Lemma 2.9. (1) *t_h is an α-one cocycle.*

(2) *Π is an unitary representation of $SL(2,\mathbb{Z})$.*

Proof. (1) $\alpha_{hk}(F) = t_{hk}\Phi(hk)^*F\Psi(hk)$, while

$$\begin{aligned}\alpha_h(\alpha_k(F)) &= \alpha_h(t_k)\Phi(k)^*\alpha_h(F)\Psi(k)\\ &= \alpha_h(t_k)\Phi(k)^*(t_h\Phi(h)^*F\Psi(h))\Psi(k)\\ &= t_h\alpha_h(t_k)\Phi(hk)^*F\Psi(hk).\end{aligned}$$

Hence we get $t_{hk} = t_h\alpha_h(t_k)$.

(2) Recall that $\Pi(h) = t_hF\alpha_h(F)^*$. Since the right-hand side is an α-one cocycle and $\alpha_k(\Pi(h)) = \Pi(h)$, we get (2). $\square$

Since $F\Psi(h)F^*\Pi(h) = \Phi(h)$ and $F\Psi(h)F^*$, $\Pi(h)$, $\Phi(h)$ are representations, we have $F\Psi(h)F^*\Pi(k) = \Pi(k)F\Psi(h)F^*$. Indeed we have

$$\begin{aligned}F\Psi(h)F^*F\Psi(k)F^*\Pi(h)\Pi(k) &= \Phi(hk) = \Phi(h)\Phi(k)\\ &= F\Psi(h)F^*\Pi(h)F\Psi(k)F^*\Pi(k).\end{aligned}$$

Hence $F\Psi(k)F^*\Pi(h) = \Pi(h)F\Psi(k)F^*$.

Lemma 2.10. *The von Neumann algebra generated by $\Pi(SL(2,\mathbb{Z}))$ is finite dimensional.*

Proof. As noted above, in fact $\Pi(SL(2,\mathbb{Z}))\subset(F\Psi(SL(2,\mathbb{Z}))F^*)'\cap(R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z})))$ and $\mathrm{Ad}\,F$ preserves $R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ globally. Hence, it is enough to show that $(\Psi(SL(2,\mathbb{Z})))'\cap(R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z})))$ is finite dimensional. Recall that $\Psi(SL(2,\mathbb{Z}))'' = \Delta(SL(2,\mathbb{Z}))'' = \{\rho_g\otimes\rho_g : g\in SL(2,\mathbb{Z})\}''$. Combining this with the fact $SL(2,\mathbb{Z})\simeq\mathbb{Z}_4 *_{\mathbb{Z}_2}\mathbb{Z}_6$, it is easy to see that $(\Psi(SL(2,\mathbb{Z})))'\cap$

$(R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z})))$ is four dimensional. Indeed, for any $x\in(\Psi(SL(2,\mathbb{Z})))'\cap(R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z})))$, consider the Fourier expansion $x=\sum_{p,q\in SL(2,\mathbb{Z})}a_{p,q}\rho_p\otimes\rho_q$. Since $((\rho_g\otimes\rho_g)x=x\rho_g\otimes\rho_g)$, we have $a_{gpg^{-1},gqg^{-1}}=a_{p,q}$. If p does not belong to the center $\mathbb{Z}_2$, its conjugacy class consists of infinite elements. Hence in this case $a_{p,q}$ must be zero. By the same way, $a_{p,q}=0$ when q does not belong to the center $\mathbb{Z}_2$. Thus, $(\Psi(SL(2,\mathbb{Z})))'\cap(R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z})))$ is four dimensional. $\square$

Lemma 2.11. *There exist unitaries $z\in L^\infty(\mathbb{T}^2)$ and $A\in R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ such that $F=z^*\otimes A$.*

Proof. Since Π is a finite-dimensional representation, we may assume that $\Pi(h_n)^*$ converges to a unitary $X\in R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ in the norm topology. Hence

$$||\alpha_{h_n}(F)-t_{h_n}XF||$$

converges to zero as $n\to\infty$. Take a spectral projection e of X such that $eX=we$ where $w\in\mathbb{T}$. (Recall that X belongs to a finite dimensional von Neumann algebra $\Pi(SL(2,\mathbb{Z}))''$.) Since e is a fixed point of α (because $e\in R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$), we get

$$||\alpha_{h_n}(eF)-t_{h_n}w(eF)||$$

converges to zero. For each normal state ρ on $R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$, we denote by T_ρ the slice map from $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ onto $L^\infty(\mathbb{T}^2)$, i.e., $T_\rho(x\otimes y)=\rho(y)x$ for $x\in L^\infty(\mathbb{T}^2)$ and $y\in R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. Obviously T_ρ commutes with α. Hence

$$||\alpha_{h_n}(T_\rho(eF))-t_{h_n}w(T_\rho(eF))||$$

converges to zero. Since eF is a non-zero element (because F is unitary), we can choose ρ such that $f=T_\rho(eF)$ is also non-zero. Next, we claim that $g=|f|$ is a constant function. Indeed, since

$$||\alpha_{h_n}(f)-t_{h_n}wf||$$

converges to zero, $||\alpha_{h_n}(g)-g||$ also converges to zero. As in the proof of Lemma 2.8, by comparing their supports as elements of $L(\mathbb{Z}^2)$, we conclude that g is constant. Thus, we may assume that f is unitary. Since both $||\alpha_{h_n}(F)-t_{h_n}XF||$ and $||\alpha_{h_n}(f)-t_{h_n}wf||$ converge to zero, $||\alpha_{h_n}(F)-(\bar{w}f^*\alpha_{h_n}(f))XF||$ and hence $||\alpha_{h_n}(f^*F)-\bar{w}X(f^*F)||$ converge to zero. Considering the supports again, we see that f^*F is a (operator-valued) constant function, i.e., $f^*F\in I\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. This means that F is of the desired form. (We take $z=f^*$.) $\square$

Combining this lemma with $\alpha_h(F) = t_h \Phi(h)^* F \Psi(h)$, we get

$$A^* \Phi(h) A = \frac{z(h^{-1}x)t(h,x)}{z(x)} \Psi(h).$$

This implies that the map $h \mapsto \frac{z(h^{-1}x)t(h,x)}{z(x)}$ is independent of the choice of x almost everywhere and define the irreducible character χ on $SL(2, \mathbb{Z})$. Hence we have $t(h,x) = \frac{z(x)\chi(h)}{z(h^{-1}x)}$.

Therefore, if we replace u by uz, we may assume that $t(h,x) = \chi(h)$, $F(h^{-1}x) = \chi(h)\Phi(h)^* F(x)\Psi(h)$ for almost all $x, y \in \mathbb{T}^2$. Indeed, we have

$$E(\tilde{\gamma}(\lambda_{\tilde{g}(h,x)})\lambda_h^*)(x) = t(h,x) = \frac{z(x)}{z(h^{-1}x)}\chi(h)$$

and hence

$$E(\tilde{\gamma}(\lambda_g)\lambda_h^*) = z\alpha_h(z)^* \chi(h)\tilde{\gamma}(f_g^{(h)}).$$

Thus

$$E(z^* u^* \gamma(\lambda_g) uz \lambda_h^*) = \chi(h)\tilde{\gamma}(f_g^{(h)}).$$

Of course uz satisfies $(uz)^* L^\infty(\mathbb{T}^2)(uz) = L^\infty(\mathbb{T}^2)$. Hence, we may assume that $t(h,x) = \chi(h)$ and $F(h^{-1}x) = \chi(h)\Phi(h)^* F(x)\Psi(h)$ for almost all $x, y \in \mathbb{T}^2$.

Lemma 2.12. *$F(x) = F(y)$ for almost all $x, y \in \mathbb{T}^2$.*

Proof. For any $\varepsilon > 0$, there exists a finite subset $\Omega \subset \mathbb{Z}^2$ such that $F_0 = \sum_{g \in \Omega} a_g \delta_g$ (a_g is an element of $R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$) and $||F - F_0||_2 < \varepsilon$ holds. We also have

$$||\alpha_{h_n}(F_0) - \chi(h_n)\Phi(h_n)^* F_0 \Psi(h_n)||_2 < \varepsilon.$$

The support of F_0 is contained in $\{(0,0)\} \cup \{\Omega \backslash (0,0)\}$ and the support of $\alpha_{h_n}(F_0)$ is contained in $\{h_n(0,0)\} \cup h_n(\{\Omega \backslash (0,0)\})$. (Note that $h_n(0,0) = (0,0)$.) Take a sufficiently large n such that

$$h_n\{\Omega \backslash (0,0)\} \cap \{\Omega \backslash (0,0)\} = \emptyset.$$

Thus, the support of $\alpha_{h_n}(F_0)$ is disjoint from $\{\Omega \backslash (0,0)\}$. Note that the support of F_0 is same as that of $\chi(h_n)\Phi(h_n)^* F_0 \Psi(h_n)$. These discussions imply that the support of F_0 is concentrated on $\delta_{(0,0)}$ with respect to $|| \cdot ||_2$. Since ε is arbitrary, $F(x)$ must be a (operator-valued) constant function. $\square$

Lemma 2.13. *There exist a unitary $w_0 \in R(SL(2,\mathbb{Z}))$, an automorphism β on $SL(2,\mathbb{Z})$ and the map $\mathbb{T}^2 \ni x \mapsto g(x) \in SL(2,\mathbb{Z})$ such that $\tilde{g}(h,x) = g(x)\beta^{-1}(h)g(h^{-1}x)^{-1}$ and $\gamma(\rho_g) = \chi(g)w_0^*\rho_{\beta(g)}w_0$.*

Proof. Since F is a (operator-valued) constant function, we have

$$\gamma^{-1}(W(x)) \otimes \gamma^{-1}(W(x))\Delta\circ\gamma^{-1}(W(x))^* = \gamma^{-1}(W(y)) \otimes \gamma^{-1}(W(y))\Delta\circ\gamma^{-1}(W(y))^*.$$

By letting $F(x,y) = \gamma^{-1}(W(y)^* W(x))$, we get

$$F(x,y) \otimes F(x,y) = \Delta(F(x,y)).$$

This implies that for almost all $x, y \in \mathbb{T}^2$, we can find the unique $g(x,y) \in SL(2,\mathbb{Z})$ such that $F(x,y) = \rho_{g(x,y)}$. Fix $x_0 \in \mathbb{T}^2$ and let $w_0 = W(x_0)$, $g(x) = g(x,x_0)$. We then have

$$\gamma^{-1}(w_0^* W(x)) = F(x_0, x) = \rho_{g(x_0,x)} = \rho_{g(x)}$$

and hence $W(x) = w_0\gamma(\rho_{g(x)})$. Combining this with $W(h^{-1}x) = \chi(h)\rho_h W(x)\gamma(\rho^*_{\tilde{g}(h,x)})$, we get

$$\gamma^{-1}\circ\mathrm{Ad}\, w_0^*(\rho_h) = \chi(h^{-1})\rho_{g(x)^{-1}\tilde{g}(h,x)g(h^{-1}x)}.$$

From this equation, we can find an automorphism β on $SL(2,\mathbb{Z})$ such that $\tilde{g}(h,x) = g(x)\beta^{-1}(h)g(h^{-1}x)^{-1}$ and $\gamma(\rho_g) = \chi\circ\beta(g)w_0^*\rho_{\beta(g)}w_0$. $\square$

The rest of the proof is now the same as that of [8]. We will include it for the sake of completeness.

Let V be the unitary of $L^\infty(\mathbb{T}^2) \otimes R(SL(2,\mathbb{Z}))$ defined by $V(x) = \chi\circ\beta(g(x))\rho_{\beta(g(x))}$ ($x \in \mathbb{T}^2$). Then by the previous lemma and $W(x) = w_0\gamma(\rho_{g(x)})$, we have $W = Vw_0$. Recall that $U = W(u_{\tilde{\gamma}} \otimes v_\gamma)$. Hence we get $U = Vw_0(u_{\tilde{\gamma}} \otimes v_\gamma) = V(u_{\tilde{\gamma}} \otimes I)w_0(I \otimes v_\gamma)$.

Compute

$$\begin{aligned} g(h^{-1}x)\beta^{-1}(h^{-1})g(x)^{-1}\sigma^{-1}x &= \tilde{g}(h,x)^{-1}\sigma^{-1}x = g(h,\sigma^{-1}x)^{-1}\sigma^{-1}x \\ &= \sigma^{-1}h^{-1}\sigma\sigma^{-1}x = \sigma^{-1}h^{-1}x \end{aligned}$$

(here we use Lemma 2.5) and get $\beta^{-1}(h^{-1})g(x)^{-1}\sigma^{-1}x = g(h^{-1}x)^{-1}\sigma^{-1}h^{-1}x$. Thus, $\beta^{-1}(h^{-1})\sigma^{-1}Tx = \sigma^{-1}Th^{-1}x$ where T is given by $Tx = \sigma g(x)^{-1}\sigma^{-1}x$. Let $S = \sigma^{-1}T$. Then we have $U = Vu_T u_{S^{-1}}w_0(I \otimes v_\gamma)$ and $\beta^{-1}(h^{-1})S = Sh^{-1}$.

Let $\omega x = \beta(g(x))x$. We would like to show $\omega = T^{-1}$. Since

$$g(\beta(g(x)^{-1}), \sigma^{-1}x) = g(x)\beta^{-1}(\beta(g(x)^{-1}))g(\beta(g(x)^{-1})^{-1}x)^{-1} = g(\beta(g(x))x)^{-1},$$

we have $g(\beta(g(x))x)\sigma^{-1}x = \sigma^{-1}\beta(g(x))\sigma\sigma^{-1}x$ and $x = \sigma g(\beta(g(x))x)^{-1}\sigma^{-1}\beta(g(x))x = T\omega x$. In particular T is surjective. Note that $\mathbb{T}^2$ is covered by the disjoint union $\bigcup_{g\in SL(2,\mathbb{Z})}\{x\in\mathbb{T}^2 : g(x) = g\}$ and T coincides with $\sigma g^{-1}\sigma^{-1}$ on $\{x\in\mathbb{T}^2\colon g(x) = g\}$. This shows that T is injective. Hence $\omega = T^{-1}$.

Thus, for $f\in L^\infty(\mathbb{T}^2)$ and $h\in SL(2,\mathbb{Z})$ we see that

$$Vu_T(f\otimes\delta_h)(x) = \chi\circ\beta(g(x))\rho_{\beta(g(x))}f(T^{-1}x)\delta_h = \chi\circ\beta(g(x))f(\beta(g(x))x)\delta_{h\beta(g(x))}.$$

Let p_g be the characteristic function of $\{x\in\mathbb{T}^2 : \beta(g(x)) = g\}$. Then for almost all x satisfying $p_g(x)\neq 0$, we get

$$Vu_T(f\otimes\delta_h)(x) = \chi\circ\beta(g)f(gx)\delta_{hg} = \chi\circ\beta(g)(p_g\otimes I)(u_g^*\otimes\rho_g)(f\otimes\delta_h)(x).$$

So we get

$$(p_g\otimes I)Vu_T = \chi\circ\beta(g)(p_g\otimes I)(u_g^*\otimes\rho_g).$$

Taking the sum with respect to g, we have

$$Vu_T = \sum_{g\in SL(2,Z)}\chi\circ\beta(g)(p_g\otimes I)(u_g^*\otimes\rho_g) = \sum_{g\in SL(2,Z)}\chi\circ\beta(g)J\pi(p_g)\lambda_g^*J.$$

Hence Vu_T commutes with $L(\mathbb{Z}^2\rtimes SL(2,\mathbb{Z}))$. Thus we have

$$\gamma = \mathrm{Ad}\,U = \mathrm{Ad}(Vu_Tu_{S^{-1}}w_0(I\otimes v_\gamma)) = \mathrm{Ad}\,u_{S^{-1}}w_0(I\otimes v_\gamma).$$

Let $v_0 = v_{\chi\circ\beta}^*v_\beta^*w_0v_\gamma$. Then, v_0 is an element of $L(SL(2,\mathbb{Z}))$ because $\mathrm{Ad}\,w_0v_\gamma(\rho_g) = \mathrm{Ad}\,w_0\gamma(\rho_g) = \chi\circ\beta(g)\rho_{\beta(g)} = \mathrm{Ad}\,v_\beta v_{\chi\circ\beta}(\rho_g)$. Then we have $u_{S^{-1}}w_0(I\otimes v_\gamma) = (u_{S^{-1}}\otimes v_\beta v_{\chi\circ\beta})v_0$ and hence $\gamma = \mathrm{Ad}(u_{S^{-1}}\otimes v_\beta v_{\chi\circ\beta})v_0$. Since $v_0\in L(SL(2,\mathbb{Z}))$ and $\beta^{-1}(h)S = Sh$ ($\Leftrightarrow S^{-1}h = \beta(h)S^{-1}$), the proof is completed. $\square$

Finally, we would like to show the claim stated in the proof of Corollary 2.2. The proof is essentially same as that of [8, Theorem 2.1]. However, since we are dealing with the non-commutative group $SL(2,\mathbb{Z})$, in order to prove the claim we need the triviality of "operator-valued eigenfunctions" on $\mathbb{T}^2$. We have already used this type of argument in the proof of Lemmas 2.8, 2.11 and 2.12.

Proof of the claim which we have postponed. Let w be a normalizer of $L(SL(2,\mathbb{Z}))$. Define $\theta = \mathrm{Ad}\,w$ and $v = w(I\otimes v_\theta^*)$. Note that $v\in L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))$. Compute

$$\begin{aligned} v(I\otimes v_\theta) = w &= J\lambda_hJwJ\lambda_h^*J \\ &= (u_h\otimes\rho_h^*)w(u_h^*\otimes\rho_h). \end{aligned}$$

Hence, we get $(I\otimes\rho_h^*)\alpha_h(v)(I\otimes\theta(\rho_h)) = v$. Then the same argument in the proof of Lemma 2.8 shows that $v\in R(SL(2,\mathbb{Z}))$. Thus $w = v(I\otimes v_\theta)\in I\otimes B(l^2(SL(2,\mathbb{Z})))$.

Combining this with the fact that w commutes with $J\lambda_h J = u_h \otimes \rho_h^*$, we see that w must belong to $L(SL(2,\mathbb{Z}))$. $\square$

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